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DIAGONALIZATION OF ABEL'S INTEGRAL OPERATOR*

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Abstract. An explicit diagonalization of Abel's integral operator

$$A(g)(y) = \int_y^\infty \frac{1}{\sqrt{x^2 - y^2}} g(x)\, dx$$

is given.

Key words. Abel integral operator, inverse problems

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1. Abel's integral equation.

Abel [1] found in 1826 that the inversion of *Abel's integral equation*

$$h(v) = \tilde{A}(g)(v) = \int_0^v \frac{1}{\sqrt{v-u}} g(u)\, du \tag{1}$$

can be given by

$$g = \tilde{A}^{-1}(h) = D\tilde{A}(h)\ ,$$

where $Dh = h'/\pi$. Because $\tilde{A}^{-2} = D$, the inversion operator $\tilde{A}^{-1}$ is a "square root" of the differentiation D and like the differentiation operator D, also $\tilde{A}^{-1}$ is unbounded. A related Abel integral equation

$$h(y) = A(g)(y) = \int_y^\infty \frac{1}{\sqrt{x^2 - y^2}} g(x)\, dx \tag{2}$$

is obtained from (1) by a change of variables $u = x^{-2}, v = y^{-2}$. The inversion formula for A is now (see [6])

$$A^{-1} = BAD$$

with the multiplication operator $B(f)(x) = -2x \cdot f(x)$. With such explicit inversion formulas, the problem seems to be solved. In applications, however, the inversion $A^{-1} = BAD$ is numerically BAD and unstable. In order to treat this so-called ill-posed problem, a better knowledge about the operator A is needed.

We report here on an inversion of A that uses an explicit diagonalization of A. This diagonalization has been designed for stellar wind tomography [5]. One might be surprised that such a diagonalization result appears only more than 160 years after Abel's work. One reason could be that usually the operator A is reduced to the operator $\tilde{A}$, which does not allow such a diagonalization: Abel obtained a singular value decomposition of $\tilde{A}$ [1, p. 100]. Our note shows that this becomes a diagonalization for the operator A.

We think that the quick inversion of A, possible by the diagonalization, might be useful from a numerical point of view. Mathematically, it is interesting to have

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a compact integral operator with explicitly known spectrum and eigenfunctions. In [2] and [3] is a large list of mathematical and physical situations in which Abel's integral operator A occurs. We recommend a consultation of these works for details and references about the different applications of A.

2. Diagonalization of Abel's integral operator. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We call a function $g : \mathbb{C}^* \to \mathbb{C}^*$ *entire at* ∞, if $z \mapsto g(z^{-1})$ is an entire function. Denote with $\mathcal{E}$ the linear space of all functions g that are entire at ∞, satisfy $g(\infty) = 0$. Every function in $\mathcal{E}$ has a Taylor development

$$g(z) = \sum_{n=1}^{\infty} g_n z^{-n},$$

which converges for all $z \in \mathbb{C}^*$.

THEOREM 2.1. *For every $n \geq 1$, the function $\psi_n(x) = x^{-n} \in \mathcal{E}$ is an eigenfunction of A with the eigenvalue*

$$\lambda_n = \int_0^{\pi/2} \cos^{n-1}(x)\, dx = \frac{\pi}{2^n \cdot n} \cdot \frac{\Gamma(n+1)}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+1}{2})} .$$

The linear operators A and A^{-1} are defined on $\mathcal{E}$ and the Taylor expansion leads to the diagonalization

$$A^{\pm 1} : \sum_{n=1}^{\infty} f_n x^{-n} \mapsto \sum_{n=1}^{\infty} \lambda_n^{\pm 1} f_n x^{-n} .$$

Proof. For $n \geq 1$, the equality

$$\int_y^{\infty} \frac{y^n}{x^n} \frac{1}{\cdot \sqrt{x^2 - y^2}}\, dx = \int_0^{\pi/2} \cos^{n-1}(\phi)\, d\phi =: \lambda_n \tag{3}$$

is verified with a substitution $\frac{y}{x} = \cos(\phi)$. Equation (3) can be rewritten as

$$A(\psi_n) = \lambda_n \cdot \psi_n$$

with $\psi_n(x) = x^{-n}$. This means that ψ_n is an eigenfunction of A with eigenvalue λ_n. It follows that for

$$f(x) = \sum_{n=1}^{\infty} f_n x^{-n} \in \mathcal{E} ,$$

the image of A or A^{-1} is

$$(A^{\pm 1}) f(y) = \sum_{n=1}^{\infty} \lambda_n^{\pm 1} f_n y^{-n} .$$

Twice a partial integration of the integral λ_n gives for $n \geq 3$ the recursion formula

$$(n-1) \cdot \lambda_n = (n-2) \cdot \lambda_{n-2}$$

for the eigenvalues. With the trivially known $\lambda_1 = \pi/2, \lambda_2 = 1$, one gets the explicit formula for the eigenvalues of A using the Euler gamma function. Because the radius of convergence of the Taylor series does not change when applying A or A^{-1}, the operator A as well as its inverse A^{-1} map $\mathcal{E}$ onto $\mathcal{E}$. $\square$

Remarks. (a) Because the formula $A^{-1} = BAD$ is easy to check for each eigenfunction using the recursion $\lambda_{n+1} = \pi(2n\lambda_n)^{-1}$, the diagonalization can be used to prove this formula for all $f \in \mathcal{E}$.

(b) The linear space $\mathcal{E}$ is a pre-Hilbert space with the scalar product

$$(f, g) = \sum_{n=1}^{\infty} f_n \bar{g}_n,$$

where $f(x) = \sum_{n=1}^{\infty} f_n x^{-n}$, $g(x) = \sum_{n=1}^{\infty} g_n x^{-n}$. Call $\mathcal{H}$ the completion of $\mathcal{E}$. The operator A can be extended to a compact bounded operator on $\mathcal{H}$ with spectrum $\{\lambda_n\}_{n=1}^{\infty}$. The inverse A^{-1} is a densely defined symmetric unbounded operator on $\mathcal{H}$ with domain $\mathcal{E}$. This domain can be enlarged to a domain $\mathcal{D}$, where A^{-1} is selfadjoint.

(c) A referee pointed out that also for $\alpha > 0$, the function $x^{-\alpha}$ is an eigenfunction of A with eigenvalue

$$\lambda(\alpha) := \int_0^{\pi/2} \cos(\phi)^{\alpha-1}\, d\phi = \frac{\pi}{2^\alpha \cdot \alpha} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\alpha+1}{2})} .$$

(See [4, p. 272] for the integral $\lambda(\alpha)$). This allows us to perform the Abel inversion on a much larger space of functions which need not to be analytic at ∞: instead of an analytic power series, one can formally transform functions of the form

$$f(x) = \int_0^{\infty} \mu(\alpha) x^{-\alpha}\, d\alpha = L(\mu)(\log(x)) ,$$

where $L(\mu)$ denotes the Laplace transform of a function or distribution $\mu = \mu(f)$. In the coordinates $\mu(f)$, the operator A is diagonal $A : \mu(f) \mapsto \lambda(\alpha) \cdot \mu(f)$. The analytic case is obtained when μ is a measure $\mu = \sum_{n=1}^{\infty} f_n \delta_n$, where δ_n denotes the Dirac measure at $n \in \mathbb{N}$.

3. Numerical example. The following Mathematica [7] procedures "AbelInverse" and "Abel" can be used to perform the Inverse-Abel and the Abel transforms for any function f. The short program does the inversion by first producing the Taylor polynomial at ∞ of a given order n.

```
ListToFunc[c_]:=Function[y,Sum[c[[k]]*y^(-k),{k,Length[c]}]];
FuncToList[f_,n_]:=Module[{q,k},q=CoefficientList[Series[f[1/y],{y,0,n}],y];
  q=Delete[q,{1}]; k=Length[q]; Do[q=Append[q,0.0],{n-k}];q];
lambda[n_]:=Table[N[Pi*2^(-i)*(1/i)*Gamma[i+1]/Gamma[(i+1)/2]^2],{i,n}];
AbelInverse[f_,n_]:=ListToFunc[1/lambda[n]*FuncToList[f,n]];
Abel[f_,n_] :=ListToFunc[lambda[n]*FuncToList[f,n]];
```

The following procedure "DataToFunc" gives a function from data by least square fitting with a polynomial of given order m.

```
DataToFunc[Data_,m_]:=Function[y,Fit[Data,Table[(1/x)^j,{j,m}],x] /. x->y];
```

Here is an example. The given data "DataExample" simulate 10 measured quantities $\{x_n, y_n\}$ for the function h. The data are fitted by a polynomial of order $m = 6$ in order to obtain a function h. Then the function $g = A^{-1}(h)$ is determined by an Abel inversion on the linear space of polynomials of order $n = 6$.

```
DataExample={{2.,0.91},{2.8,0.49},{3.6,0.35},{4.4,0.27},{5.2,0.37},
             {6.,0.54},{6.8,0.46},{7.6,0.25},{8.4,0.13},{9.2,0.10}};
h=DataToFunc[DataExample,6];
g=AbelInverse[h,6];
```

In concrete applications, one has to refine the strategy according to the situation. The fitting and the choice of the orders of the polynomials can be especially crucial. We have done the case of wind tomography in [5].

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REFERENCES

[1] N. H. Abel, *Résolution d'un problème de mecanique*, J. Reine Angew. Math. Bd I, 1826; See *Oevres complètes de Niels Henrik Abel*, p. 97–101, L. Sylow and S. Lie, eds.

[2] R. Anderson and F. Hoog, *Abel Integral Equations*, in Numerical Solution of Integral Equations, M. Golberg, ed., Plenum Press, New York, 1989.

[3] I. Craig and J. Brown, *Inverse Problems in Astronomy, a guide to inversion strategies for remotely sensed data*, Adam Hilger, Bristol and Boston, 1986.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1980.

[5] O. Knill, R. Dgani, and M.Vogel, *A new approach to Abel's integral operator and its application to stellar winds*, Astron. and Astrophys, 274:1002:1008, 1993, p. 1002.

[6] G. Minerbo and M. Levy, *Inversion of Abel's Integral equation by means of orthogonal polynomials*, SIAM J. Numer. Anal. 6, (1969), p. 598.

[7] Wolfram Research, *Mathematica Version* 2.0, Addison–Wesley, Illinois, 1991.